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Published in:
Journal of Mathematical Analysis and Applications

Publication date:
1997

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **212**, 417–429 (1997)
ARTICLE NO. AY975514

A Better Triangulation for Wright's 2^n-Ray Algorithm*

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Submitted by Harlan W. Stech

Received March 20, 1995

We give a triangulation for Wright's 2^n-ray algorithm to compute solutions of nonlinear equations. According to measures of efficiency of triangulations, it is better than any other available triangulation for the 2^n-ray algorithm.

1. INTRODUCTION

In [27] Wright proposed a variable dimension simplicial algorithm to compute solutions of nonlinear equations. It is called a 2^n-ray algorithm. This type of algorithm was originated by Scarf in [21]. The subsequent developments of it can be found in [2, 4, 5, 7, 11–14, 16, 18, 22, 27, 28] and so on. For a survey on simplicial algorithms, see, e.g., [1]. We set $N = \{1, 2, \ldots, n\}$. For any $m \in N$ and $t \in \{-1, 1\}$, we define $W(m, t)$ by

$$W(m,t) = \left\{x \in R^n \mid |x_j| \le tx_m,\ j \ne m\right\}.$$

Clearly, the union of all $W(m,t)$'s is equal to R^n. The 2^n-ray algorithm needs a triangulation of R^n, the restriction of which on $W(m,t)$ is a triangulation of $W(m,t)$ for any $m \in N$ and $t \in \{-1, 1\}$. The K_1-triangu-

*The authors thank the anonymous referee for the remarks and comments.

0022-247X/97 $25.00

lation in [9] and J_1-triangulation in [23] are such available triangulations. Theoretical results and numerical experience have shown that the efficiency of simplicial algorithms depends heavily on the triangulations underlying them. For details, see [6, 8, 17, 23–25] and so on. In [3] the D_1-triangulation was proposed. According to measures of efficiency of triangulations, it is better than the K_1-triangulation and J_1-triangulation. However, the D_1-triangulation is not suitable for use in the 2^n-ray algorithm. In this paper we give a triangulation for the 2^n-ray algorithm, which is a modification of the D_1-triangulation. We call it the D_{v2}-triangulation. It has the same properties as the D_1-triangulation. Thus it is better than any other available triangulation for the 2^n-ray algorithm. Although there are a few other attractive triangulations of R^n in [10, 15, 19, 20, 26], etc., it is complicated to obtain from them triangulations for use in the 2^n-ray algorithm.

The paper is organized as follows. We give the D_{v2}-triangulation in Section 2. We describe pivot rules of the D_{v2}-triangulation in Section 3. We discuss properties of the D_{v2}-triangulation in Section 4.

2. D_{v2}-TRIANGULATION

Throughout this paper we assume $2 \leq n$. When a triangulation of $W(1,1)$ is available, it is not difficult to obtain from it a triangulation of R^n, the restriction of which on $W(m,t)$ is a triangulation of $W(m,t)$ for any $m \in N$ and $t \in \{-1, 1\}$. Therefore, as follows, we only give a triangulation of $W(1,1)$.

We set $D = \{y \in W(1,1) \mid \text{components of } y \text{ are even}\}$. Let y be a vector of D. For a given y, we define $I(y)$ and $J(y)$ by

$$I(y) = \{i \in N \mid y_1 = |y_i|\} \qquad \text{and} \qquad J(y) = \{j \in N \mid y_1 > |y_j|\}.$$

Let $s = (s_1, s_2, \ldots, s_n)^\top$ be a sign vector such that if $y_1 = 0$ then $s_1 = 1$, and that, for any $i \in I(y)$ with $i \neq 1$, when $s_1 = -1$, $s_i = -1$ if $y_i > 0$, and $s_i = 1$ if $y_i < 0$. We define $K(y,s) = \{i \in I(y) \mid s_i y_i = y_1\}$. Let l denote the number of elements in $I(y)$ and h the number of elements in $K(y,s)$. Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of elements of N such that, for r with $\pi(r) = 1$, if $h = 0$ then $j > r$ for $j \neq r$ with $\pi(j) \in I(y)$ and if $h > 0$ then $j < r$ for $j \neq r$ with $\pi(j) \in K(y,s)$. Let p be an integer with $0 \leq p \leq n-1$ satisfying that if $h = 0$ then $\{\pi(k) \mid p \leq k \leq n\} \neq I(y)$ and if $h > 0$ then $\{\pi(k) \mid p \leq k \leq n\} \neq \{\pi(k) \in K(y,s) \mid p \leq k \leq n\}$. When $h = 0$, for $j = 1, 2, \ldots, n$, we define $g(j) =$

$(g_1(j), g_2(j), \ldots, g_n(j))^\top$ by, if $j = 1$,

$$g_i(j) = \begin{cases} s_i & \text{if } i \in I(y), \\ 0 & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n$, and by, if $j \neq 1$,

$$g_i(j) = \begin{cases} s_i & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n$. When $h > 0$, for $j = 1, 2, \ldots, n$, we define $g(\pi(j)) = (g_1(\pi(j)), g_2(\pi(j)), \ldots, g_n(\pi(j)))^\top$ by, if $\pi(j) \in K(y, s)$,

$$g_i(\pi(j)) = \begin{cases} s_i & \text{if } i \in K(y, s) \text{ and } j \leq \pi^{-1}(i), \\ 0 & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n$, and by, if $\pi(j) \notin K(y, s)$,

$$g_i(\pi(j)) = \begin{cases} s_{\pi(j)} & \text{if } i = \pi(j), \\ 0 & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n$. Let u^i be the ith unit vector of R^n for $i = 1, 2, \ldots, n$.

DEFINITION 1. For any y, π, s, and p as above, $y^0, y^1, \ldots, y^n$ are given as follows.

If $p = 0$ then $y^0 = y$ and

$$y^k = y + g(\pi(k)), \qquad k = 1, 2, \ldots, n.$$

If $p \geq 1$ then $y^0 = y + s$ and

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, p - 1,$$

$$y^k = y + g(\pi(k)), \qquad k = p, \ldots, n.$$

Let $y^0, y^1, \ldots, y^n$ be as above. Clearly, they are affinely independent. Thus their convex hull is a simplex. We denote it by $D_{v2}(y, \pi, s, p)$. Let D_{v2} be the set of all such simplices. We illustrate D_{v2} for $n = 3$ in Fig. 1. As follows, we show that D_{v2} is a triangulation of $W(1, 1)$.

From Definition 1, it is not difficult to check that, for any σ^1 and σ^2 of D_{v2}, the intersection of σ^1 and σ^2, $\sigma^1 \cap \sigma^2$, is either empty or a common face of both of them. Obviously, for any point of $W(1, 1)$, there is a neighborhood of it, which meets a finite number of simplices of D_{v2}. Therefore, to demonstrate that D_{v2} is a triangulation of $W(1, 1)$, we only need to show that the union of all simplices of D_{v2} is equal to $W(1, 1)$.

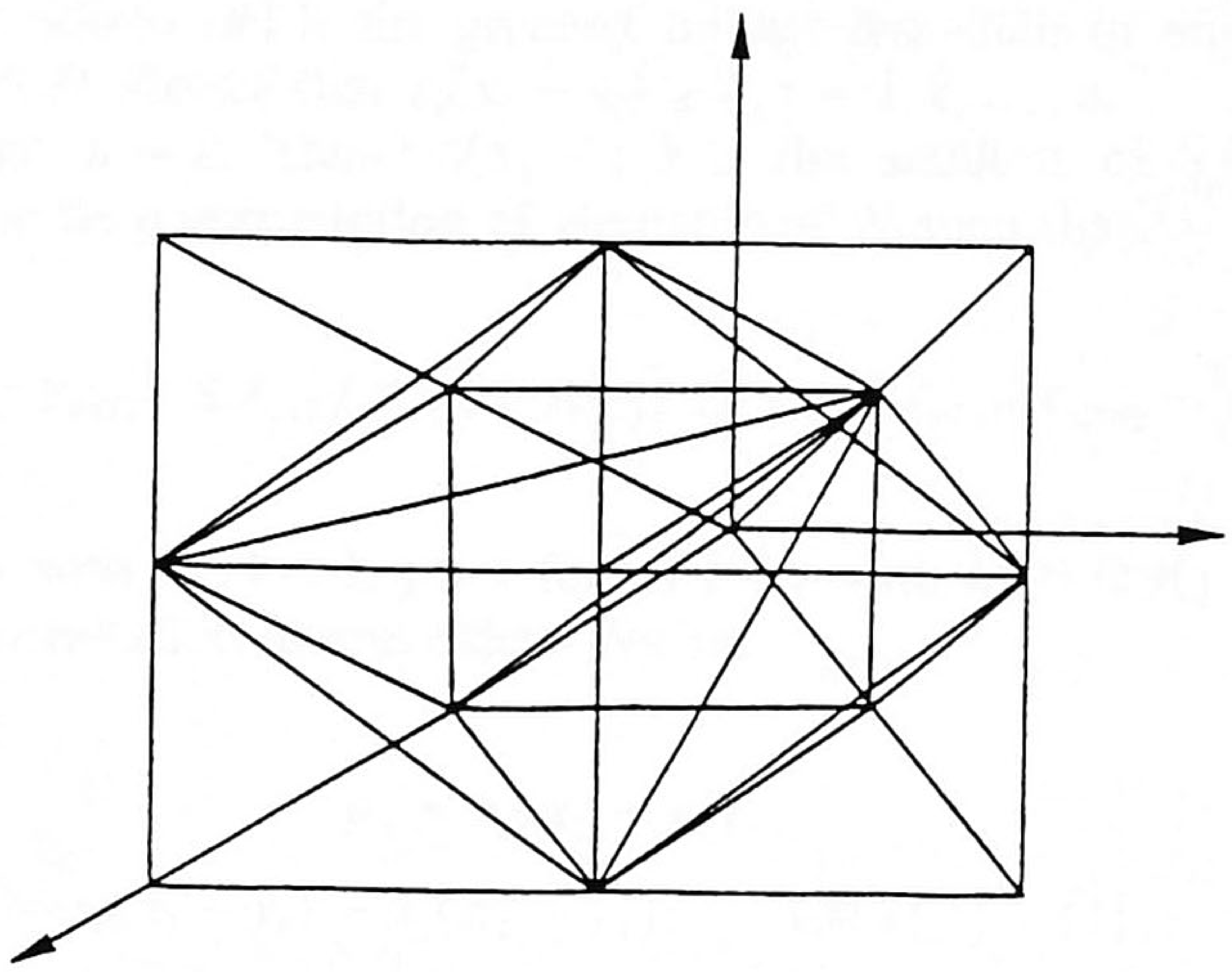

FIG. 1. D_{v2}-triangulation of $W(1,1)$ for $n = 3$.

For any given y, π, s, and p as above, we define q by

$$q = |K(y,s) \cap \{\pi(k) \mid 1 \le k < p\}|,$$

and let $\pi(i_1), \pi(i_2), \dots, \pi(i_h)$ with $i_1 < i_2 < \cdots < i_h$ denote elements in $\{\pi(k) \in K(y,s) \mid 1 \le k \le n\}$. When $\pi(i) \in K(y,s)$ and $i > i_{q+1}$, we set $i_{-1} = i_{k-1}$, where k is the index with $i = i_k$.

LEMMA 1. *The union of all simplices of D_{v2} is equal to $W(1,1)$, i.e.,*

$$\cup_{\sigma \in D_{v2}} \sigma = W(1,1).$$

Proof. Clearly, every simplex in D_{v2} lies in $W(1,1)$. Let x be an arbitrary point of $W(1,1)$. We show $x \in D_{v2}(y, \pi, s, p)$ with y, π, s, p determined as follows. Let $y = (y_1, y_2, \dots, y_n)^\top$ be given by

$$y_i = \begin{cases} \lfloor x_i \rfloor & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ \lfloor x_i \rfloor + 1 & \text{otherwise,} \end{cases}$$

$i = 1, 2, \dots, n$, and let $s = (s_1, s_2, \dots, s_n)^\top$ be given by

$$s_i = \begin{cases} 1 & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ -1 & \text{otherwise,} \end{cases}$$

$i = 1, 2, \dots, n$, where $\lfloor \theta \rfloor$ is the greatest integer less than or equal to θ. Obviously, $y \in D$. Notice that $s_i(x_i - y_i) \geq 0$, $i = 1, 2, \dots, n$.

Suppose that $h = 0$. Then $s_1(x_1 - y_1)$ is the smallest of $s_i(x_i - y_i)$, $i \in I(y)$. Let π be a permutation of elements of N such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}),$$

and that, for r with $\pi(r) = 1$, $j > r$ for all $j \neq r$ with $\pi(j) \in I(y)$. Notice that such a permutation always exists. We set

$$\mu_1 = s_1(x_1 - y_1),$$
$$\mu_i = s_i(x_i - y_i) - s_1(x_1 - y_1), \qquad i \in I(y) \setminus \{1\},$$
$$\mu_j = s_j(x_j - y_j), \qquad j \in J(y).$$

Let $\mu = \sum_{j=1}^{n} \mu_j$.

Suppose that $\mu \leq 1$. We set $p = 0$. Let $\beta_0 = 1 - \mu$, and $\beta_i = \mu_{\pi(i)}$, $i = 1, 2, \dots, n$. Clearly, $\beta_i \geq 0$, $i = 0, 1, \dots, n$, and $\sum_{i=0}^{n} \beta_i = 1$. From Definition 1, the vertices of $D_{v2}(y, \pi, s, p)$ are given by $y^0 = y$ and

$$y^k = y + g(\pi(k)), \qquad k = 1, 2, \dots, n.$$

It is not difficult to check that $x = \sum_{j=0}^{n} \beta_j y^j$. Thus, $x \in D_{v2}(y, \pi, s, p)$.

Suppose that $\mu > 1$. Let $p_{\max}$ denote the largest integer p with $1 \leq p \leq n - 1$ such that $\{\pi(k) \mid p \leq k \leq n\} \neq I(y)$. We show that there exists some integer p, with $2 \leq p \leq p_{\max}$ for $l = n$ and $1 \leq p \leq p_{\max}$ for $l < n$, such that the following β_j, $j = 0, 1, \dots, n$, are nonnegative,

$$\beta_0 = s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),$$
$$\beta_1 = s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),$$
$$\dots$$
$$\beta_{p-2} = s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) - s_{\pi(p-2)}(x_{\pi(p-2)} - y_{\pi(p-2)}),$$
$$\beta_{p-1} = -s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) + \lambda_p,$$
$$\beta_k = s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - \alpha_k, \qquad k = p, p+1, \dots, n,$$

where

$$\lambda_p = \begin{cases} \dfrac{\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1}{n-p} & \text{if } r < p, \\ \dfrac{\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1 - (l-1)s_1(x_1 - y_1)}{n-p-l+1} & \text{otherwise}, \end{cases}$$

and

$$\alpha_k = \begin{cases} \lambda_p & \text{if } r < p \text{ or both } r \geq p \text{ and } \pi(k) \notin I(y) \setminus \{1\}, \\ s_1(x_1 - y_1) & \text{if } r \geq p \text{ and } \pi(k) \in I(y) \setminus \{1\}, \end{cases}$$

$k = p, p+1, \ldots, n$.

If $\beta_{p-1} \geq 0$ for $p = p_{\max}$, then $\beta_k \geq 0$, $k = 0, 1, \ldots, n$, and we set $p = p_{\max}$. Suppose that $\beta_{p-1} < 0$ for $p = p_{\max}$. Since $\mu > 1$, there exists some integer p_0, with $2 \leq p_0 \leq p_{\max} - 1$ for $l = n$ and $1 \leq p_0 \leq p_{\max} - 1$ for $l < n$, such that

$$0 \leq -s_{\pi(p_0-1)}(x_{\pi(p_0-1)} - y_{\pi(p_0-1)}) + \lambda_{p_0}$$

and either both $r = p_0 + 1$ and $p_0 = n - l$ or

$$0 > -s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) + \lambda_{p_0+1}.$$

Thus,

$$s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) - \alpha_{p_0} \geq 0.$$

Hence, when we set $p = p_0$, $\beta_k \geq 0$, $k = 0, 1, \ldots, n$. Clearly, $\sum_{k=0}^{n} \beta_k = 1$. From Definition 1, the vertices of $D_{v2}(y, \pi, s, p)$ are given by $y^0 = y + s$ and

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, p-1,$$
$$y^k = y + g(\pi(k)), \qquad k = p, \ldots, n.$$

It is not difficult to check that $x = \sum_{j=0}^{n} \beta_j y^j$. Thus, $x \in D_{v2}(y, \pi, s, p)$.

Suppose that $h > 0$. Let π be a permutation of elements of N such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \leq \cdots \leq s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}),$$

and that, for r with $\pi(r) = 1$, $j < r$ for all $j \neq r$ with $\pi(j) \in K(y, s)$. Notice that such a permutation always exists. Let $\mu_1 = s_1(x_1 - y_1)$ and $\mu_i = s_i(x_i - y_i)$, $i \in N \setminus K(y, s)$. We set $\mu = \mu_1 + \sum_{i \in N \setminus K(y,s)} \mu_i$.

Suppose that $\mu \leq 1$. We set $p = 0$. Let $\beta_0 = 1 - \mu$, and

$$\beta_i = \begin{cases} s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) \\ \quad - s_{\pi(i_{-1})}(x_{\pi(i_{-1})} - y_{\pi(i_{-1})}) & \text{if } \pi(i) \in K(y, s) \text{ and } i > i_1, \\ s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n$. Clearly, $\beta_j \geq 0$, $j = 0, 1, \ldots, n$, and $\sum_{j=0}^{n} \beta_j = 1$. From Definition 1, the vertices of $D_{v2}(y, \pi, s, p)$ are given by $y^0 = y$ and

$$y^k = y + g(\pi(k)), \qquad k = 1, 2, \ldots, n.$$

It is not difficult to check that $x = \sum_{j=0}^{n} \beta_j y^j$. Thus, $x \in D_{v2}(y, \pi, s, p)$.

Suppose that $\mu > 1$. Let $p_{\max}$ denote the largest integer p with $1 \leq p \leq n - 1$ such that $\{\pi(k) \mid p \leq k \leq n\} \neq \{\pi(k) \in K(y, s) \mid p \leq k \leq n\}$. We show that there exists some integer p with $1 \leq p \leq p_{\max}$ such that the following β_j, $j = 0, 1, \ldots, n$, are nonnegative,

$$\begin{aligned} \beta_0 &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ \beta_1 &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ &\cdots \\ \beta_{p-2} &= s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) - s_{\pi(p-2)}(x_{\pi(p-2)} - y_{\pi(p-2)}), \\ \beta_{p-1} &= -s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) + c(p), \\ \beta_i &= s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) - \nu_i, \qquad i = p, p+1, \ldots, n, \end{aligned}$$

where

$$c(p) = \begin{cases} \dfrac{\sum_{j=p}^{n} s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - 1}{n - p} & \text{if } r < p, \\ \dfrac{\sum_{j=p}^{n} \rho_{\pi(j)} - 1}{n - p - h + q + 1} & \text{otherwise}, \end{cases}$$

with

$$\rho_{\pi(j)} = \begin{cases} 0 & \text{if } \pi(j) \in K(y, s) \text{ and } i_{q+1} \leq j < i_h, \\ s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) & \text{otherwise}, \end{cases}$$

$j = p, p + 1, \ldots, n$, and where

$$\nu_i = \begin{cases} s_{\pi(i_{-1})}(x_{\pi(i_{-1})} - y_{\pi(i_{-1})}) & \text{if } i > i_{q+1} \text{ and } \pi(i) \in K(y, s), \\ c(p) & \text{otherwise}, \end{cases}$$

$i = p, p + 1, \ldots, n$.

If $\beta_{p-1} \geq 0$ for $p = p_{\max}$, then $\beta_k \geq 0$, $k = 0, 1, \ldots, n$, and we set $p = p_{\max}$. Otherwise, since $\mu > 1$, there exists some integer p_0 with $1 \leq p_0 \leq p_{\max} - 1$ such that

$$0 \leq -s_{\pi(p_0-1)}(x_{\pi(p_0-1)} - y_{\pi(p_0-1)}) + c(p_0)$$

and

$$0 > -s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) + c(p_0+1).$$

Thus,

$$s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) - \nu_{p_0} \geq 0.$$

Hence, when we set $p = p_0$, $\beta_k \geq 0$, $k = 0, 1, \ldots, n$. Obviously, $\sum_{j=0}^{n} \beta_j = 1$. From Definition 1, the vertices of $D_{v2}(y, \pi, s, p)$ are given by $y^0 = y + s$ and

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, p-1,$$
$$y^k = y + g(\pi(k)), \qquad k = p, \ldots, n.$$

It is not difficult to check that $x = \sum_{j=0}^{n} \beta_j y^j$. Thus $x \in D_{v2}(y, \pi, s, p)$.

From the above discussions, the lemma follows immediately. ∎

Now we obtain the following conclusion.

THEOREM 1. *D_{v2} is a triangulation of $W(1,1)$.*

We call it the D_{v2}-triangulation of $W(1,1)$.

3. PIVOT RULES OF THE D_{v2}-TRIANGULATION

To use the D_{v2}-triangulation in the 2^n-ray algorithm, we need its pivot rules, which show how to move from a simplex to its adjacent simplices. Let $\sigma = D_{v2}(y, \pi, s, p)$ be a simplex of the D_{v2}-triangulation of $W(1,1)$ with vertices $y^0, y^1, \ldots, y^n$. In the case that the facet of σ opposite to y^i doesn't lie in the boundary of $W(1,1)$, we want to obtain the simplex $\bar{\sigma} = D_{v2}(\bar{y}, \bar{\pi}, \bar{s}, \bar{p})$, which shares the facet with σ. We show how to obtain $\bar{y}, \bar{\pi}, \bar{s}$, and $\bar{p}$ from y, π, s, p, and i in Table I, where j^*, $p^\#$, π^*, and p^* are given as follows: $j^* = \pi(k)$ with $k \neq i$, $\pi(k) \neq 1$, and $n-2 \leq k \leq n$. If $\pi(n-1) = 1$, $s_{\pi(n-1)} = -1$, and $\pi(k) \in I(\bar{y})$, $k = n-1, n$, then $p^\# = p - 1$. If $\pi(k) \in K(\bar{y}, \bar{s})$, $k = n-1, n$, then $p^\#$ is the smallest integer such that $\pi(j) \in K(\bar{y}, \bar{s})$, $j = p^\# + 1, \ldots, n$. Otherwise, $p^\# = p$. If $\pi(n) = 1$, $s_{\pi(n)} = -1$, and $\pi(k) \in I(\bar{y})$, $k = n-1, n$, then $\pi^* =$

$(\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$ and $p^* = p - 1$. If $\pi(k) \in K(\bar{y}, \bar{s})$, $k = n-1, n$, then $\pi^* = (\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$ and p^* is the smallest integer such that $\pi(j) \in K(\bar{y}, \bar{s})$, $j = p^* + 1, \ldots, n$. Otherwise, $\pi^* = \pi$ and $p^* = p$. In Table I, BD means that the facet of σ opposite to y^i lies in the boundary of $W(1,1)$.

4. PROPERTIES OF THE D_{v2}-TRIANGULATION

Let H be a subset of N with $1 \in H$, and $V = \{v_j \in \{-1, 1\} \mid j \in H \setminus \{1\}\}$. For any given H and V, we set $C(H, V) = \{x \in W(1,1) \mid x_1 = v_j x_j, j \in H \setminus \{1\}\}$. Clearly, $C(H, V)$ is a face of $W(1,1)$. We can treat x_j, $j \in H$, as one component. Therefore, we can obtain the D_{v2}-triangulation of $C(H, V)$ from Definition 1. Similarly to the K_1-triangulation and J_1-triangulation of $W(1,1)$, the restriction of the D_{v2}-triangulation of $W(1,1)$ on $C(H, V)$ gives a triangulation of $C(H, V)$, which is the same as the D_{v2}-triangulation of $C(H, V)$. Notice that when a unit cube is entirely contained in $W(1,1)$, the restriction of the D_{v2}-triangulation on it is the same as the D_1-triangulation. The number of simplices of the D_1-triangulation in an n-dimensional unit cube is equal to

$$N(D_1) = 2 + \sum_{k=2}^{n-1} \frac{n!}{(n-k+1)!}.$$

The number of simplices of the K_1-triangulation or J_1-triangulation in an n-dimensional unit cube is equal to

$$N(K_1) = N(J_1) = n!.$$

We know that $N(D_1) < N(K_1)$ for $3 \leq n$, and $N(D_1)/N(K_1)$ approaches to $e - 2$ as n goes to infinity. It implies that the number of simplices of the D_{v2}-triangulation is asymptotically about $e - 2$ times that of the K_1-triangulation or J_1-triangulation. In addition, the diameter and average directional density of the D_{v2}-triangulation are as good as those of the K_1-triangulation or J_1-triangulation, or even better since the D_{v2}-triangulation is almost the same as the D_1-triangulation. Therefore, the D_{v2}-triangulation is better than the K_1-triangulation and J_1-triangulation according to measures of efficiency of triangulations.

According to numerical tests we have done, as a triangulation for the 2^n-ray algorithm, the D_{v2}-triangulation is indeed better than the K_1-triangulation or J_1-triangulation.

TABLE I

i	p			
0	0	$h = 0$	$l = n$	$n = 2$
				$n \geq 3$
			$l < n$	
		$h > 0$	$h = n$	
			$h < n$	
	1			
	$p \geq 2$			
$1 \leq i \leq n$	0	$h = 0$	$\pi(i) \in J(y)$ or $\pi(i) = 1$	
			$\pi(i) \in I(y)$, $\pi(i) \neq 1$	
		$h > 0$	$\pi(i) \in K(y, s)$	
			$i_1 < i < \pi^{-1}(1)$	
			$i_1 < i = \pi^{-1}(1)$	
			$i = i_1$ or $\pi(i) \notin K(y, s)$	
$1 \leq i < p - 1$		$h = 0$	$\pi(i) = 1$	$\pi(i + 1) \in I(y)$
				$\pi(i + 1) \notin I(y)$
			$\pi(i) \neq 1$	
		$h > 0$	$\pi(i) \in K(y, s)$	$\pi(i + 1) = 1$
				$\pi(i + 1) \neq 1$
			$\pi(i) \notin K(y, s)$	
$i = p - 1$	$p \geq 2$	$h = 0$	$\pi(k) \in I(y)$, $k = p, \ldots, n$	
			$\pi(i) = 1$	
			otherwise	
		$h > 0$		
$p \leq i \leq n$	$1 \leq p < n - 1$	$h = 0$	$\pi^{-1}(1) < p$	
			$\pi(i) \in I(y)$	$\pi(i) \neq 1$
			$\pi^{-1}(1) \geq p$	$\pi(i) = 1$
			$\pi(i) \notin I(y)$	$p = n - 2$, $\pi(j) \in I(y)$
			$\pi^{-1}(1) \geq p$	$j = p, \ldots, n$, $j \neq i$
				$p < n - 2$, $\pi(j) \in I(y)$
				$j = p, \ldots, n$, $j \neq i$
				$\pi(j) \notin I(y)$ for
				some $p \leq j \leq n$, $j \neq i$
		$h > 0$	$\pi(i) \in K(y, s)$	$i_{q+1} < i < \pi^{-1}(1)$
				$i_{q+1} < i = \pi^{-1}(1)$
				$i = i_{q+1}$
			$\pi(i) \notin K(y, s)$	$\pi(j) \in K(y, s)$
				$j = p, \ldots, n$, $j \neq i$
				$\pi(j) \notin K(y, s)$ for
				some $p \leq j \leq n$, $j \neq i$
	$1 \leq p = n - 1$		$i = n - 1$	
			$i = n$	

TABLE I—*Continued*

$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
$y + 2s_{\pi(n)}u^{\pi(n)}$	$s - 2s_{\pi(n)}u^{\pi(n)}$	π	1
y	s	π	2
y	s	π	1
$y + 2s_{\pi(n)}u^{\pi(n)}$	$s - 2s_{\pi(n)}u^{\pi(n)}$	π	$n - 1$
y	s	π	1
y	s	π	0
y	$s - 2s_{\pi(1)}u^{\pi(1)}$	π	p
y	$s - 2s_{\pi(i)}u^{\pi(i)}$	π	p
BD			
y	s	$(\pi(1), \ldots, \pi(i_{-1} - 1), \pi(i), \pi(i_{-1} + 1), \ldots, \pi(i - 1), \pi(i_{-1}), \pi(i + 1), \ldots, \pi(n))$	p
BD			
y	$s - 2s_{\pi(i)}u^{\pi(i)}$	$(\pi(i), \pi(1), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	p
BD			
y	s	$(\pi(1), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$	p
y	s	$(\pi(1), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$	p
BD			
y	s	$(\pi(1), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$	p
y	s	$(\pi(1), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$	p
y	s	π	$p - 2$
y	s	π	$p - 1$
y	s	π	$p - 1$
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
BD			
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
$y + 2s_{j^*}u^{j^*}$	$s - 2s_{j^*}u^{j^*}$	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 2$
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
y	s	$(\pi(1), \ldots, \pi(i_{-1} - 1), \pi(i), \pi(i_{-1} + 1), \ldots, \pi(i - 1), \pi(i_{-1}), \pi(i + 1), \ldots, \pi(n))$	p
BD			
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
$y + 2s_1u^1$	$s - 2s_1u^1$	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$n - 1$
y	s	$(\pi(1), \ldots, \pi(p - 1), \pi(i), \pi(p), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n))$	$p + 1$
$y + 2s_{\pi(n)}u^{\pi(n)}$	$s - 2s_{\pi(n)}u^{\pi(n)}$	π	$p^{\#}$
$y + 2s_{\pi(n-1)}u^{\pi(n-1)}$	$s - 2s_{\pi(n-1)}u^{\pi(n-1)}$	π^*	p^*

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